Preference for Consumption Predictability and the Equity Premium Puzzle*

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Abstract

This paper provides a solution for the equity premium puzzle. We modify the standard constant relative risk aversion utility function by assuming that the representative consumer also has a preference for consumption predictability. While keeping the conditional mean of the stochastic discount factor close to one, this feature not only reinforces consumption smoothing, but it also results in large increases in the variability of the stochastic discount factor which is crucial for this solution to the puzzle. The large increase in variability for the stochastic discount factor in the modified model is primarily determined by large, realized consumption forecast errors. Although these oversized forecast errors arise infrequently, when they do arise, they result in very high aversion to risk and enhanced interest in smoothing consumption.

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1 Introduction

Since Mehra and Prescott (1985) pointed out the difficulties of standard consumption-based intertemporal economic models for explaining key financial market features, the so called equity premium puzzle has been a long standing issue in the economics and finance literature. For the most part, there are two categories for research on the equity premium puzzle. First, many researchers study data to document that the equity premium does exist and what its behavior looks like over time.[1] Second, other researchers focus on how to rationalize the equity premium in variations of the standard consumption-based intertemporal economic model used in macroeconomics and finance. Among others, Cochrane (2001) and Mehra (2003) provide outstanding critical reviews of alternative theoretical approaches suggested for finding a solution to this puzzle. Mehra (2003) organizes and discusses many of these important extensions of the standard consumption-based intertemporal economic model and explains why they fail to solve the equity premium puzzle. These extensions include; (i) alternative assumptions about preferences, as in Abel (1990), Benartzi and Thaler (1995), Campbell and Cochrane (1999), Constantinides (1990), and Epstein and Zin (1991); (ii) the presence of incomplete markets, as in Mankiw (1986), Constantinides and Duffie (1996), Heaton and Lucas (1997) and Storesletten, Telmer, and Yaron (2007); (iii) market imperfections, as in Aiyagari and Gertler (1991) and Alvarez and Jermann (2000); (iv) alternative probability distributions to allow for rare, but disastrous events, as in Rietz (1988); and (v) the hypothesis of the survivorship bias, suggested in Brown, Goetzmann, and Ross (1995).

Subsequent to the Cochrane (2001) and Mehra (2003) reviews, additional papers

[1] Some papers, such as Bessler (1999), Mehra, (2006), Vivian (2007), Sarkar and Zhang (2009), Rieger, Hens and Wang (2013), Horvath (2020) and Christou, Gupta and Jawadi (2021), study the equity premium in markets other than the United States. Others, such as Cogley (2002), Park, (2006), Shackman, (2006), Sarkar and Zhang (2009), Sarantis and Ekaterini (2013), Ma (2013), Jacobs, Pallage and Robe (2013), Kim (2016), Smith (2017), Avdis and Wachter (2017), Bonaparte and Fabozzi (2017) and Wilson (2020), use novel econometric methods or data sets to document the equity premium. While others, such as Lettau, Ludvigson and Wachter (2008) and Wachter and Warusawitharana (2015), investigate how the equity premium changes over time.

have further contributed to the theory rationalizing the equity premium.[2] In evaluating any theoretical solution to the puzzle, it is important to keep in mind Mehra (2003) which convincingly argues that, "the equity premium puzzle is a quantitative puzzle": while the standard consumption-based intertemporal model and its extensions are aligned with the qualitative risk prediction that stocks should return more than bonds on average, the puzzle really shows up because the quantitative predictions from most of these extended models fall short by a wide margin in explaining the wedge between stock and bond returns.

This paper suggests an alternative theoretical solution to the equity premium puzzle. We propose a behavioral extension of the standard constant relative aversion utility function in which the representative agent not only shows a preference for smoothing consumption in the standard way, through a relatively low intertemporal elasticity of substitution (IES), but also shows a preference for predictability of consumption. This assumption of a preference for consumption predictability introduced here is very much related to a theoretical and empirical literature in which social scientists from different fields (economics, psychology, and sociology) have suggested that discrepancies between achievements and expectations affects individuals' subjective well-being (among others, Campbell, 1976; Mason and Faulkenberry, 1978; Higgins, 1987; Bertoni and Corazzini, 2018).

The desire for predictability of consumption creates an additional smoothing feature beyond the risk aversion and intertemporal substitution properties implied in the standard model. This additional feature not only enhances the desire for consumption smoothing, but also has implications for the stochastic discount factor, the rate of relative risk aversion (RRA) and the IES. We show that this feature results

[2] Most of these papers build on some of the earlier models. For instance, Melino and Yang (2003), Allais (2004), Zeisberger, Langer, and Trede (2007), Fielding and Stracca (2007), Heiberger (2020) and Fujii, and Nakamura (2021) consider alternative preference structures. Others, such as Jermann (2010), Gollier and Schlee (2011), Favilukis (2013), Dunbar (2013), Wilson (2020) and Kim (2021), consider market structures in which agents are heterogenous and participate in the risk markets in different ways or markets are incomplete somehow. While others, such as Cogley and Sargent (2008), Julliard and Ghosh (2012), Suzuki (2014), Wang and Mu (2019), Horvath (2020) consider extensions of the rare events idea.

in a stochastic discount factor that is considerably more variable than the one implied by standard preferences when confronted with annual times series consumption data from 1955 to 2022, and this increased volatility in the stochastic discount factor solves the puzzle. Our solution considers an alternative preference structure like many other proposed solutions, but it also has a connection to the rare events literature because rare events in the observed data are important for generating the equity premium. However, one distinction from the other rare events solutions suggested in the literature is that the rare events in this study are not so rare that they have never been observed. Instead, the rare events identified in the analysis with our modified preferences are somewhat extreme events featuring large consumption forecast errors that have been observed during recent economic times, such as the financial crises prior to the Great Recession, or the Covid pandemic.[3]

Our primary empirical objective was to find parameters that result in a Hansen-Jagannathan bound that is consistent with observed values. We find there are many such parameters. Furthermore, in an effort to provide insight into the empirical results, we decompose the stochastic discount factor into some of its components to reveal the source for the increased variability. This source is also connected to the RRA and IES, which are shown to be time varying in the modified model. The observed consumption series is generally smooth and consumption forecast errors are generally modest. However, the model implies that even modest consumption forecast errors result in higher volatility in the stochastic discount factor, a higher RRA and a lower IES. The observed consumption series also exhibits periods in which the consumption forecast errors can be very large, such as during steep economic contractions. During these times, the stochastic discount factor can spike up. In addition, during these periods when the consumption series becomes less predictable and the consumption forecast errors become large, the consumer's desire for consumption smoothness is enhanced and can be seen in a higher RRA and lower IES.

[3] Although their focus was not so much on the equity premium puzzle, Cogley and Sargent (2008) also use an observed rare event, the Great Depression, in their analysis.

In general, agents can be understood to want to hold lower risk assets, despite the equity premium, because the loss in utility when consumption becomes unpredictable is large and painful. A secondary empirical investigation was also undertaken which sought parameters to match the comovement between the stochastic discount factor and risky returns. For this investigation, the parameters which fit this comovement were similar to those found to fit the Hansen-Jagannathan bound. Taken together, it was shown that several important empirical observations can be simultaneously achieved to resolve the equity premium puzzle.

2 Consumer preferences with consumption forecast errors

The most common utility function used to demonstrate the equity premium puzzle is the constant rate of relative risk aversion (CRRA) utility function. This utility function is popular in the macroeconomics literature, with well known properties, and because of this, makes for a useful starting point. Various insights into the puzzle can be seen with this utility function and these insights have motivated various modifications in an effort to resolve the puzzle. We follow a similar strategy here.

One of the insights from the CRRA utility function notes that for reasonable levels of risk aversion, observed consumption is too smooth for observed asset returns (Mehra and Prescott, 1985). Alternatively, one can say that the observed smooth consumption data and the large risky asset returns can only be reconciled with huge levels of risk aversion in a standard CRRA utility function. We use this insight to motivate a modified utility function in which agents prefer further smoothing for consumption than in the standard CRRA utility function, thus making the desire for smooth consumption line up better with the observed data at sound values of the risk aversion parameter. To achieve this, we add to the CRRA utility function a term that reflects a preference for predictability of consumption. Here, in addition to obtaining utility from consumption, agents predict future consumption using information they have available at a particular date, and the closer their prediction is to observed consumption, the higher their utility, while the worse their prediction, the worse their utility. We summarize these desires by assuming consumers make consumption choices so as to maximize

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\gamma} c_t^{1-\gamma} - \frac{\mu}{1-\gamma} \left(\omega c_t^2 + (c_t - E_{t-1}\{c_t\})^2 \right)^{\frac{1-\gamma}{2}} \right) \right\},$$

where c_t is consumption at date t, and subscripts on the expectations indicate information sets available at the date of the subscript. We often refer to this utility function as the modified utility function in our discussion below.[4]

The utility function has four parameters: the subjective discount factor, β, the standard relative risk aversion parameter, γ, and the two additional parameters, μ and ω, describing the new term capturing the taste for consumption predictability. The following restrictions are placed on each of the parameters: $0 < \beta < 1$, $\gamma > 0$, $\mu \geq 0$ and $\omega > 0$. The first two parameter restrictions are consistent with the standard assumptions seen in the CRRA utility function, while the assumption that μ is nonnegative reflects the assumption that forecast errors result in lower utility. Here we allow the possibility that $\mu = 0$ in order to allow the standard CRRA utility function to be a special case of this utility function. We restrict ω to be strictly positive so as to avert any potential problems that might arise should a perfect forecast of consumption be obtained and thus the terms in the second part of the utility function equal zero and are then raised to a potentially negative power (this would be the case if $\gamma > 1$). Finally, the values of μ and ω have additional restrictions that they cannot be too large to ensure that the marginal utility of consumption will always be non negative as is assumed in any standard characterization of consumer preferences.

Some further comments about the second term of the utility function are useful for highlighting three important properties of this utility function specification: scale invariance, symmetry for positive and negative forecast errors, and an interaction between the marginal utility of consumption and (the size of) forecast errors.

[4] Because information sets change over time, these preferences are time varying, which is a type of state dependence, but differs from Melino and Yang (2003).

Scale invariance: The two interior terms of the second part of the utility function are each raised to a power of 2 to insure that a situation in which a negative number is raised to a power does not occur. Because c_t will always be positive, raising it to a power of 2 is not essential. However, the forecast error term need not be positive, so raising it to a power of 2 is essential to avoid powers of a negative number arising. Because the forecast error is raised to a power of 2, this induces the power of 2 on the c_t term as well as the 2 in the denominator of the exponent $\frac{1-\gamma}{2}$. These additional power restrictions are used in order to ensure that the modified utility function is also scale invariant as the standard CRRA utility function, which is a useful property that allows the utility function to be used in settings with growing arguments.

Symmetry for positive and negative forecast errors: Raising the forecast error to a power of 2 in the utility function implies a symmetry for positive and negative forecast errors. This means that perfectly predictable consumption yields the highest utility and missed predictions lead to lower utility with the same utility reduction for overforecasting and underforecasting by equal amounts. This implies that agents have a second motivation for smoothing consumption. In addition to the standard smoothing motivation due to the intertemporal elasticity of substitution, agents also desire consumption to be predictable. This assumption of a preference for consumption predictability is very much related to the theoretical and empirical literature in which social scientists from different fields have suggested that discrepancies between achievements and expectations affects individuals' subjective well-being (see Bertoni and Corazzini, 2018; and references therein).

Marginal utility of consumption depends on forecast errors: Including the positive term ωc_t^2 not only overcomes the potential problem, noted above, associated with the second part of the utility function being zero, but also results in the marginal utility of consumption depending on consumption forecast errors. It can be shown that the marginal utility of consumption is in general a decreasing function of consumption forecast errors.[5]

[5] A supplementary appendix provides some additional details for the mathematical expressions shown in the paper, including a demonstration that the marginal utility of consumption is a decreas-

Finally, one additional notational definition will often be used. In many cases it will be useful to use the notation $\varepsilon_t = c_t - E_{t-1}\{c_t\}$ to indicate the forecast error at t for forecasts of c_t based on information up to time $t-1$. This results in some simplification to the second term of the utility function and will make some of the expressions below more friendly.

2.1 Agent optimization and asset pricing relationships

Using this utility function, one can derive some common expressions used in asset pricing investigations. Assuming a consumer budget constraint which includes both a risky and risk-free asset, and using standard optimization methods, an intertemporal first order condition relating asset returns to utility is given by

$$E_t \left\{ \frac{\beta \left[c_{t+1}^{-\gamma} - \mu \left(\omega c_{t+1}^2 + (\varepsilon_{t+1})^2 \right)^{\frac{1-\gamma}{2}-1} \left[\omega c_{t+1} + \left((\varepsilon_{t+1})^2 \right)^{\frac{1}{2}} \right] \right] R_{t+1}}{\left[c_t^{-\gamma} - \mu \left(\omega c_t^2 + (\varepsilon_t)^2 \right)^{\frac{1-\gamma}{2}-1} \left[\omega c_t + \left((\varepsilon_t)^2 \right)^{\frac{1}{2}} \right] \right]} \right\} = 1,$$

where R_{t+1} is the return on an asset.[6] To solidify notation, we will leave R_{t+1} without any further notation to indicate the return on the risky asset and we will add a superscript f to indicate a risk-free asset return, as in R_{t+1}^f. In the asset pricing literature it is common to define the marginal rate of substitution between time t goods and time $t+1$ goods by

$$m_{t+1} = \frac{\beta \left[c_{t+1}^{-\gamma} - \mu \left(\omega c_{t+1}^2 + (\varepsilon_{t+1})^2 \right)^{\frac{1-\gamma}{2}-1} \left[\omega c_{t+1} + \left((\varepsilon_{t+1})^2 \right)^{\frac{1}{2}} \right] \right]}{\left[c_t^{-\gamma} - \mu \left(\omega c_t^2 + (\varepsilon_t)^2 \right)^{\frac{1-\gamma}{2}-1} \left[\omega c_t + \left((\varepsilon_t)^2 \right)^{\frac{1}{2}} \right] \right]},$$

and to reduce the first order condition to $E_t \{m_{t+1} R_{t+1}\} = 1$.[7] A further set of standard algebraic steps can be implemented to obtain the Hansen-Jagannathan bound

ing function of the forecast errors.

[6] We use the expression $\left((\varepsilon_{t+1})^2 \right)^{\frac{1}{2}}$, rather than $|\varepsilon_{t+1}|$, to be clear that various functions are differentiable because absolute value expressions can sometimes be unclear as to their differentiability.

[7] m_{t+1} also goes by other names including, stochastic discount factor and pricing kernel.

(Hansen and Jagannathan, 1991) given by

$$\left| \frac{E_t\left(R_{t+1}\right) - R^f_{t+1}}{std_t\left(R_{t+1}\right)} \right| \leq \frac{std_t\left(m_{t+1}\right)}{E_t\left(m_{t+1}\right)}, \tag{1}$$

where the term on the left side of the inequality is known as the Sharpe Ratio and the term on the right is the Hansen-Jagannathan bound.

Several insights about the Hansen-Jagannathan bound can be seen from these expressions. For instance, notice that a risk-free asset return must also satisfy the first order condition $E_t\left\{m_{t+1}\right\} R^f_{t+1} = 1$, which implies that $E_t\left\{m_{t+1}\right\} \approx 1$ since observed risk-free asset returns are close to one.[8] This implies that the Hansen-Jagannathan bound is roughly equal to $std_t\left(m_{t+1}\right)$. Furthermore, studying the expression for the stochastic discount factor, m_{t+1}, one can easily see that in the standard CRRA model (i.e. when $\mu = 0$), its conditional volatility, $std_t\left(m_{t+1}\right)$, roughly equals the product of the parameter γ times the growth rate of consumption, c_{t+1}/c_t. Hence, for the observed Sharpe ratio and consumption growth values, combined with any sound, low value of gamma, the Hansen-Jagannathan inequality (1) does not hold, which results in the equity premium puzzle found in the standard model. However, when agents care about consumption predictability, m_{t+1} also depends on consumption forecast errors, helping to increase the size of the conditional volatility of m_{t+1} for any given values of γ and growth rate of consumption while keeping the conditional expectation of m_{t+1} close to one as discussed below. In short, the modified utility function solves the equity premium puzzle and the risk-free rate puzzle at once.

Consumption forecast errors increase the stochastic volatility of m_{t+1} through two channels. First, realized consumption forecast errors enter in the denominator of m_{t+1} (i.e. the current marginal utility of consumption) lowering it, and because the current marginal utility of consumption is known at time t it works as scale factor in the definition of the conditional stochastic volatility of m_{t+1}. That is, the larger is μ and/or the forecast error, $(\varepsilon_t)^2$, the lower is the denominator of m_{t+1} and the larger its stochastic volatility. Second, the stochastic volatility of m_{t+1} also depends

[8] Mehra (2003, p. 58, Table 4) reports a value of 1.008.

on the volatility of the numerator of m_{t+1}, which increases with the volatility of consumption forecast errors.

For our primary empirical exercises below, we will find it is also useful to use the expression obtained one step prior to (1), which is given by

$$\frac{E_t\left(R_{t+1}\right)-R_{t+1}^f}{std_t\left(R_{t+1}\right)}\times\frac{-1}{\rho_{m,R}}=\frac{std_t\left(m_{t+1}\right)}{E_t\left(m_{t+1}\right)}, \tag{2}$$

where $\rho_{m,R}$ denotes the conditional correlation between R_{t+1} and m_{t+1}.[9]

2.2 Intertemporal relationships

In order to gain intuition on the role of the additional terms entering in the stochastic discount factor, m_{t+1}, in explaining the equity premium puzzle it is useful to obtain expressions for the rate of relative risk aversion (RRA) and the intertemporal elasticity of substitution (IES) associated with the modified model. These concepts are given by the following expressions:

$$RRA_t = \frac{\gamma c_t^{-\gamma-1}-\mu\left[\left(\gamma+1\right)\Omega_t^{\frac{1-\gamma}{2}-2}\left[\omega c_t+\left(\left(\varepsilon_t\right)^2\right)^{\frac{1}{2}}\right]^2-\Omega_t^{\frac{1-\gamma}{2}-1}(\omega+1)\right]}{c_t^{-\gamma}-\mu\Omega_t^{\frac{1-\gamma}{2}-1}\left[\omega c_t+\left(\left(\varepsilon_t\right)^2\right)^{\frac{1}{2}}\right]}\times c_t$$

and

$$ISE_{t+1} = \frac{\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}-\mu\widetilde{\Omega}_{t+1}^{\frac{-\gamma-1}{2}}\left[\omega\left(\frac{c_{t+1}}{c_t}\right)+\left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]\right]}{\left[\gamma\left(\frac{c_{t+1}}{c_t}\right)^{-(\gamma+1)}-\mu(\gamma+1)\widetilde{\Omega}_{t+1}^{\frac{-\gamma-1}{2}-1}\left[\omega\left(\frac{c_{t+1}}{c_t}\right)+\left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]^2+\mu(\omega+1)\widetilde{\Omega}_{t+1}^{\frac{-\gamma-1}{2}}\right]}\times\left(\frac{c_{t+1}}{c_t}\right)^{-1},$$

where we use the notations $\Omega_t=\left(\omega c_t^2+\left(\varepsilon_t\right)^2\right)$ and $\widetilde{\Omega}_{t+1}=\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)$ to condense the expressions so they will fit on the page.

A few comments about the RRA_t are insightful. First, when $\mu=0$, the RRA_t equals γ. This follows because, as noted earlier, the standard CRRA utility function is just a special case of the modified utility function used here. Furthermore, like in

[9] An alternative fitting exercise is discussed in Section 3.4 which uses a slight variation on this expression.

the CRRA utility function, the denominator is the marginal utility of consumption, which is the same denominator as in m_{t+1}. This fact will be important later when we interpret the empirical results. Finally, unlike the CRRA utility function, when $\mu \neq 0$, the RRA_t is time varying which is why we have included a time subscript on it.

For the ISE_t some additional comments are also insightful. First, as in the CRRA utility function, $RRA_t = 1/ISE_t$ for all values of γ, μ and ω.[10] This means that ISE_t is also time varying. This is in clear contrast with Epstein and Zin's (1991) preferences where the two concepts are determined by two different utility parameters.

3 Empirical results

An algebraic analysis of the determinants of the Hansen-Jagannathan bound under the modified utility function is somewhat challenging because the stochastic discount factor, m_{t+1}, adds additional terms to the standard stochastic discount factor that are highly non-linear. As a way of overcoming this issue, in this section we empirically investigate the implications of the modified utility function using a statistical model of the per-capita consumption series, based on annual U.S. data from 1955 to 2022, to characterize the equity premium.[11] A similar approach was followed by Reis (2009) to characterize the costs of aggregate fluctuations.[12] By using a statistical model rather than a fully-articulated dynamic stochastic general equilibrium model, we are following an agnostic approach that allows us to assess whether it is feasible to reconcile aggregate consumption and return data with the basic pricing equation associated with a sound characterization of preferences that include a preference for consumption predictability. This approach abstracts from any other additional features that characterize any particular dynamic stochastic general equilibrium model

[10] A proof of this is available in the supplementary appendix.

[11] In a robustness investigation described later, we use quarterly data from 1954:q3 to 2023:q1.

[12] As emphasized by Reis (2009, footnote #1), a statistical model can be interpreted as an economic model of an endowment economy in which the path of consumption is described by a statistical model of observed data on consumption.

one might considered, and thus focuses only on the ability of the modified utility function for explaining the equity premium puzzle. The fitting approach is a calibration type approach which is popular in the equity premium puzzle literature. Here we make use of some commonly used statistical numbers and show that it is possible to find parameter values that are consistent with the observed data and these common statistical values. Because most of the equity premium puzzle literature focuses on annual data, our primary investigation focuses on annual data for the period 1955-2022 which was aggregated from quarterly data, but in the robustness section we also investigate quarterly data and another annual sample period.

Following most of the related literature, we begin our analysis by focusing on the relationship between the Hansen-Jagannathan bound and the Sharpe ratio. Later, in Section 3.4, we will extend our analysis to study the equity premium from a different angle which looks directly at the optimality condition $E_t\left\{m_{t+1}R_{t+1}\right\}=1$.

The key statistic to match is the Sharpe ratio ($\frac{E_t(R_{t+1})-R^f_{t+1}}{std_t(R_{t+1})}$). A common value for annual data is 0.37, but some researchers use values as large as 0.50, so we focus on this range.[13] We confirm this empirical range for the Sharpe ratio value in Table 1 by considering data at both annual and quarterly frequencies over several periods of time. Table 1 also shows that the Sharpe ratio is lower for quarterly data than annual data, which is a well known fact, and is often mentioned as showing that the equity premium puzzle is even worse for long-horizon investors and long-horizon returns.[14] A common value for the correlation between the stochastic discount factor and the return on risky assets ($\rho_{m,R}$) is -0.20.[15] So, our exercise is to show that it is possible to find values for the model parameters in the modified model so that the Hansen-Jagannathan bound ($\frac{std_t(m_{t+1})}{E_t(m_{t+1})}$) is in the range of 1.85 to 2.50 as needed to satisfy (2).

Some of the utility function parameters are set or restricted to commonly used values or ranges. For instance, we set $\beta=0.96$, which is a value often used with

[13] Mehra (2003) uses the value 0.37, while Cochrane (2001) considers the value of 0.50.
[14] See for instance, Cochrane (2001, p.462).
[15] This is the value noted by Cochrane (2001, p. 457).

annual data.[16] We also, focus on a range of γ values given by 1.0 to 1.90, which is a reasonable range according to the empirical literature and for which the CRRA utility function is unable to match the empirical data for.[17,18] The remaining two parameters, μ and ω, are then set to achieve the Hansen-Jagannathan bound. It turns out, there are many, so we only provide a few in our discussion below. In particular, in our primary calibration we use ω equal to 0.1 and search over the μ range until the Hansen-Jagannathan bound was achieved.[19]

Table 1. Sharpe ratio

	Annual	Quarterly
Sample period		
1934-2022	0.57	
1934:q1-2023:q1		0.30
1955-2022	0.52	
1954:q3-2023:q1		0.28

Notes: Stock and bond return data used in the computation of the Sharpe ratio was downloaded on June 28, 2023 from Robert Shiller's website. This is the updated data set used in Robert Shiller's book "Irrational Exuberance" Princeton University Press, 2000, 2005, 2015, editions.

3.1 Fitting based on U.S per-capita consumption

To compute the Hansen-Jagannathan bound implied by the modified utility function, we need to compute values for the conditional expectation and conditional standard deviation of m_{t+1}. This requires several steps. First, we need to compute a series for m_{t+1}. For this we need a per-capita consumption series and a per-capita consumption

[16] In our robustness section, we also fit the model to quarterly data. For that exercise we use $\beta = 0.99$.

[17] We actually use a value of $\gamma = 0.99$ as the lower end of the range to avoid the extra coding required if γ precisely equals 1.

[18] For instance, Smets and Wouters (2007) estimates a 5%−95% posterior credible set of $(1.16, 1.59)$ using post-WWII US data. Typical papers addressing the equity premium puzzle find that γ needs to be unusally large (Mehra and Prescott, 1985).

[19] In exercises not reported here, we were also able to fit the data with very small ω values such as 1.0×10^{-7} as well as larger values up to 1.0.

forecast error. To get these, we compute U.S. real per-capita consumption using standard formulas.[20] Next, we use this data in a forecasting equation to obtain the forecasting errors. We investigated several forecasting models, and use a random walk model in our primary investigation. This model was chosen in part because it is often found that per-capita consumption has a unit root, but also because it was among the best models among the many we investigated according to standard criteria such as Akaike and Schwarz-Bayesian information criteria.[21] From this forecast model we generate a series for the forecast errors given by $\varepsilon_t = c_t - E_{t-1}\{c_t\}$. With the per-capita consumption series and the forecast error series, we then compute m_{t+1} for a particular parameter setting.

Next, we need to compute conditional expectations and conditional standard deviations for m_{t+1}. To do this, we run a generalized autoregressive conditional heteroskedastic model with one autoregressive term and one moving average term ($GARCH(1,1)$) and collect from this model conditional values for the forecast and variance. We then average these conditional forecasts to get an estimate for $E_t\left(m_{t+1}\right)$ and average the conditional variances to obtain an estimate for $Var_t\left(m_{t+1}\right)$ which are then used to compute the Hansen-Jagannathan bound.

Computation of the Hansen-Jagannathan bound is computed for many different parameter settings and these computations are then plotted in figures such as Figure 1 below to provide intuition and insight into the implications of the modified utility function. Looking at the left hand edge of Figure 1, where $\mu = 0$, which is the value in which the modified utility function equals the CRRA utility function, we see that the Hansen-Jagannathan bound is too small. This is the typical finding seen in the equity premium literature. In particular, for values of γ in the empirical range, the

[20] Nominal aggregate consumption time series is defined by (quarterly, seasonally adjusted annual rate) Personal Consumption Expenditures (PCEC), expressed in billions of US Dollars. Real aggregate consumption is obtained deflating the previous consumption time series by (quarterly, seasonally adjusted) implicit GDP price deflator index. Finally, real per-capita consumption is obtained dividing real agregate consumption by an index of the population level based on the population time series: CNP16OV. Price and population indexes consider the year 2012 as the base year. All time series were retrieved from FRED: Federal Reserve Economic Data on June 17th, 2023.

[21] In the robustness section we also discuss forecasting models which use AR(1) and AR(2) forecasting models for differenced per-capita consumption.

Hansen-Jagannathan bound never gets close to 0.37, the lower value of the range for the Sharpe ratio reported in Mehra (2003). However, for values of μ that are not too large, we see that it is possible to get Hansen-Jagannathan bounds in the range of 1.85 to 2.5 (i.e. the range of Hansen-Jagannathan bound values consistent with the Sharpe ratio estimates of 0.37 and 0.5, respectively, and the conditional correlation between stock returns and discount factors of -0.2).[22] These values are seen in the back corner of Figure 1. Furthermore, these parameter value combinations of γ and μ are not unique, and it is possible for any and all of the parameters to be altered by small amounts and still produce a figure that is qualitatively similar to Figure 1.

3.2 Insights into the larger Hansen-Jagannathan bound

Understanding the source of the larger Hansen-Jagannathan bound is fairly straight forward. Looking at (2), one sees that the larger bound can occur by either a smaller $E_t\left(m_{t+1}\right)$ or a larger $std_t\left(m_{t+1}\right)$. However, as noted above, the value of $E_t\left(m_{t+1}\right)$ is restricted to be close to one based on the average of observed risk-free asset returns, so the difference needs to arise from the $std_t\left(m_{t+1}\right)$ term. We find that for both the standard CRRA utility function and the modified utility function the values for $E_t\left(m_{t+1}\right)$ are similar, but the modified results in a larger $std_t\left(m_{t+1}\right)$. The larger $std_t\left(m_{t+1}\right)$ arises mostly because of the difference in the denominator for m_{t+1} associated with the modified utility function. The standard CRRA specification implies a denominator of $c_t^{-\gamma}$ while the modified specification has a denominator of $c_t^{-\gamma} - \mu\left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_t + \left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]$. Both of these are the marginal utility of consumption for the respective specifications. When $\mu = 0$, the two denominators are equal and the Hansen-Jagannathan bounds are the same and are not very large as seen on the left front edge of Figure 1. However, as μ increases, the second term in the denominator of the modified specification starts to approach the value of the first term in the denominator making the overall denominator approach zero. Of

[22] The value of $\rho_{m,R} = -0.20$ assumed here is the value suggested in Cochrane (2001, p. 457). We will relax this assumption by examining directly the covariance between the stochastic discount factor and the risky returns below.

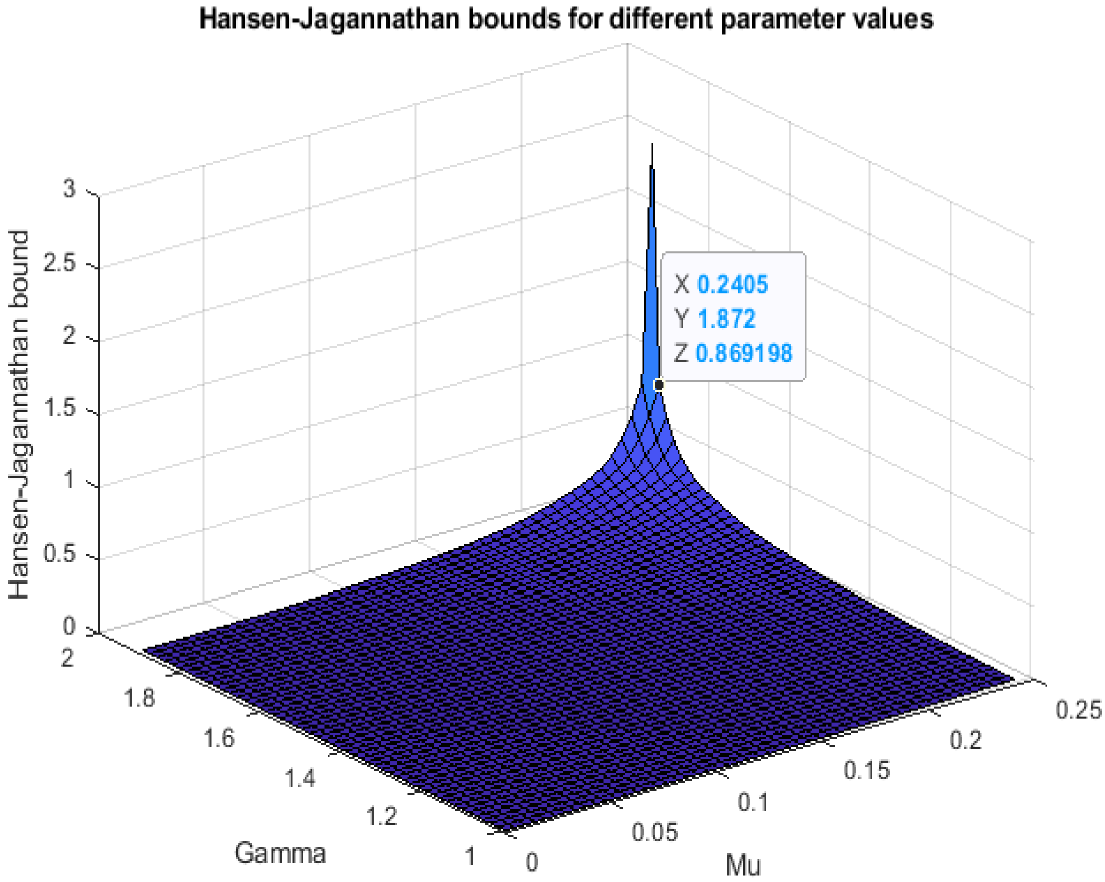

Figure 1: Hansen-Jagannathan bound grid

course other parameters play a role, such as the value of ω, γ and β, so the value of μ that induces the volatility varies depending on these other parameters, but the easiest way to understand the essence of the increased m_{t+1} volatility is to see that increasing μ results in the denominator moving nearer to zero and thus creating the larger percentage standard deviation seen on the right side of (2).

This increased volatility in m_{t+1} can also be understood empirically by studying Figures 2, 3 and 4 below. These figures correspond to the calibration at the shoulder value marked in the Figure 1 where the Hansen-Jagannathan bound is only 0.87. We choose to use this calibration rather than the peak calibration where the Hansen-Jagannathan bound is 2.5 because the volatility of m_{t+1} at the peak value of the modified specification is so large that the variability of the standard CRRA volatility of m_{t+1} is not noticeable in the diagram.

In Figure 2, the blue line shows the empirical values for m_{t+1} for the standard CRRA model when $\gamma = 1.872$ and $\beta = 0.96$ using the observed consumption data, while the orange line shows the empirical values for m_{t+1} for the modified model using the same γ and β values and $\mu = 0.24$ and $\omega = 0.1$. Simple inspection shows that the mean of the empirical values for m_{t+1} are roughly equal, but the variance and hence the standard deviations are much different with the modified model having a much larger variance. Together, this similar mean but larger variance induce the larger percentage standard deviation in m_{t+1} in the modified specification, which results in the achievement of the Hansen-Jagannathan bound.

Figure 3 shows the source for this volatility increase. The blue line in Figure 3 plots the observed values for $c_t^{-\gamma}$ while the orange line plots the observed values for $\mu\left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_t + \left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]$. Both series decline over time because there is an upward trend in c_t and the exponent -1.872 (i.e. $-\gamma$) induces the decline. The blue line has two uses. First, the blue line represents the denominator for m_{t+1} in the standard CRRA model, but it also represents the first term in the denominator

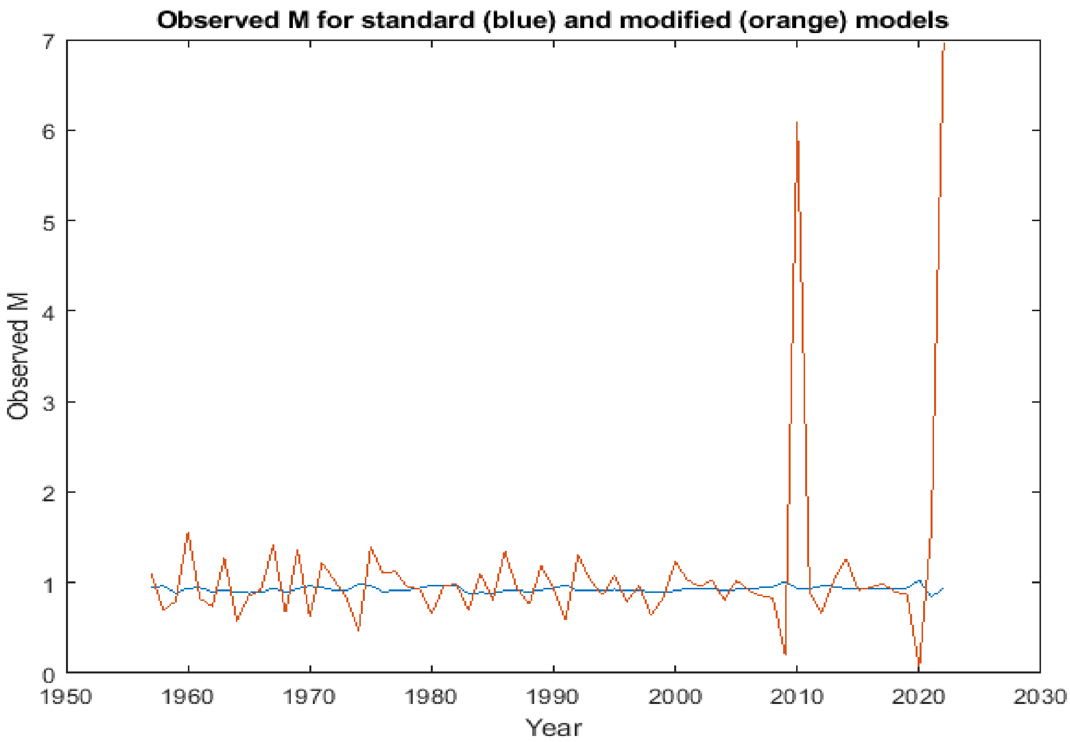

Figure 2: The stochastic discount factor

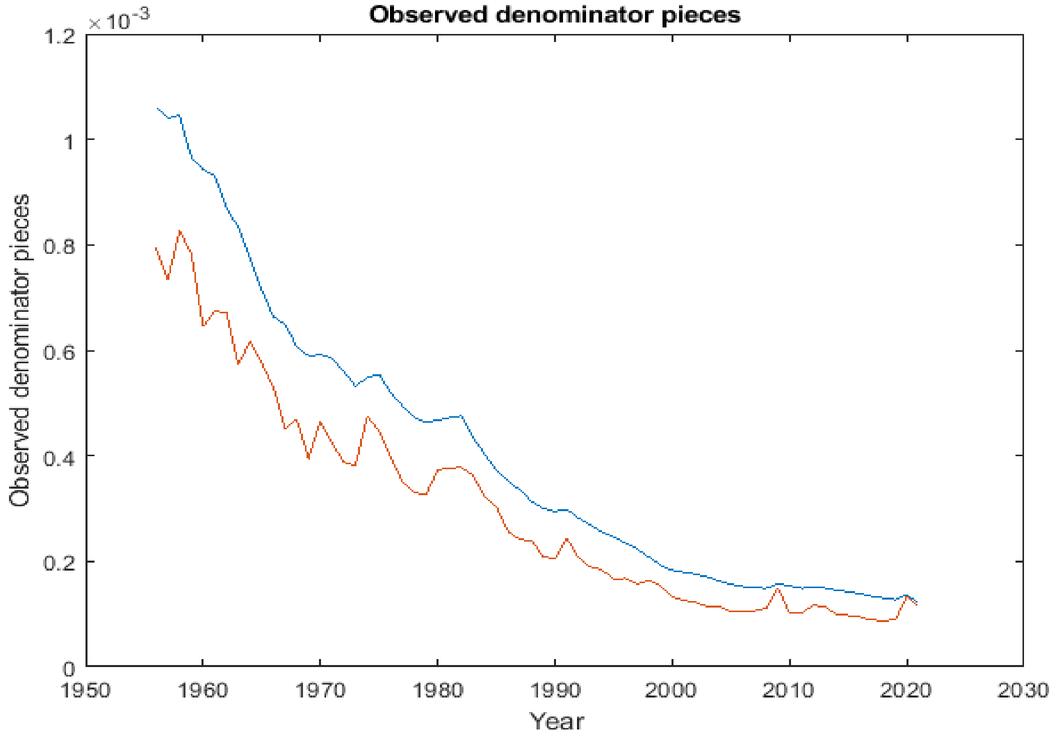

Figure 3: Denominator pieces of the stochastic discount factor

for the modified model. The orange line plots the second term in the denominator of the modified model and, as can be seen, sometimes the lines suddenly become very close, inducing a term near zero in the denominator of m_{t+1}. For instance, during the Global Financial Crises as well as during the Covid Recession there were sharp swings in c_t, inducing large consumption forecast errors as shown in Figure 4, and thus large fluctuations in m_{t+1}. These large fluctuations in m_{t+1} during the Global Financial Crises and the Covid Recession can be seen in Figure 2.

Figure 4 also shows two interesting features of the data which have important implications for the conditional variance in m_{t+1}. First, the large forecast errors around the Global Financial Crises and the Covid Recession are important drivers of the high volatility of m_{t+1}, and they consequently have a large impact on the value of the Hansen-Jagannathan bound seen in Figure 1. This means our findings are somewhat aligned with Rietz (1988), which solved the equity premium puzzle based on disaster states. However, although the modified model also includes extreme outcomes to solve the puzzle, the magnitudes are not as extreme as required by Rietz (1988). The modified specification only requires consumption outcomes that have been witnessed in the historical data, such as those that occurred during the Global Financial Crises and the Covid Recession. Furthermore, the value of γ need not be unusually large. Rietz (1988) on the other hand, requires the prospect of a 1 in 100 chance of a 25 percent decline in consumption to reconcile the equity premium with a rather high risk-aversion parameter of 10 (Mehra, 2003). Second, another interesting feature of the data is that although consumption forecast errors appear to have a constant mean, they do appear to be heteroskedastic with volatility varying over time.

One final set of insights into the mechanism for the larger Hansen-Jagannathan bound can be obtained by considering RRA_t and ISE_t. Recall that the denominator for m_{t+1} and RRA_t are the same. So, as seen in Figure 3, the two denominators can become very small when consumption forecast errors become large. It is during these periods in which the forecast errors are large that the rate of relative risk aversion becomes very high and agents become strongly inclined to hold less risky assets in order to induce consumption smoothing. This desire for consumption smoothing can also be seen by recalling that $RRA_t = 1/ISE_t$ which implies that as RRA_t rises, ISE_t falls, again implying that agents become more inclined to smooth consumption not only across states of nature as indicated by a high RRA_t, but also over time as indicated by a low ISE_t. Although these comments are made for a particular calibration on a particular set of observed data, the time variation in the RRA_t and ISE_t occurs in general (recall that $RRA_t = 1/ISE_t$ for all values of γ, μ and ω). This means that regardless of the data, with this modified utility function, agents

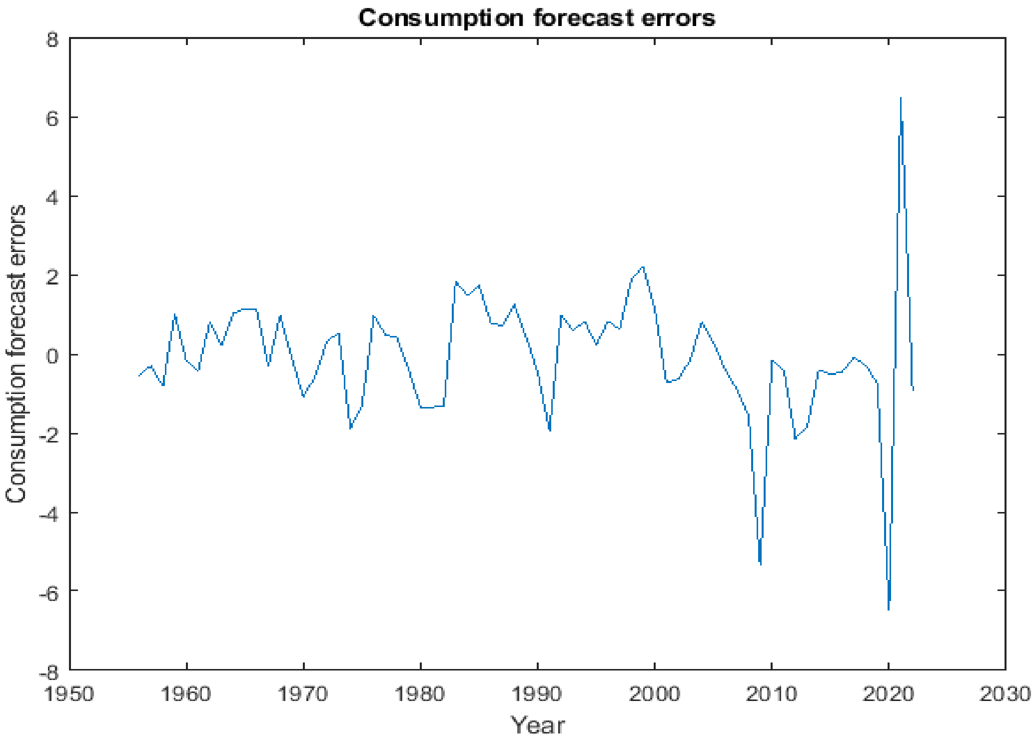

Figure 4: Consumption forecast errors

desire to hold a relatively low risk portfolio despite an equity premium because large movements from smooth consumption imply large consumption forecast errors and consequently high utility loses, higher aversion to risk and a stronger desire to not substitute consumption over time.

3.3 Robustness

We considered a number of alternative settings to those discussed in our primary investigation above in order to assess the robustness of these results. Some of these exercises were to use the same data but different values of ω to see if a calibration could be obtained that would attain the Hansen-Jagannathan bound. As noted earlier, the value of ω was flexible, and for many values we were able to get the desired Hansen-Jagannathan bound. Other exercises included two different consumption forecasting models. In a supplementary appendix that is available from the authors upon request, we show that figures that are very similar to Figure 1 can be obtained when an AR(1) or an AR(2) model to forecast differenced consumption is used. Also in that appendix, it is shown that the results are robust to changes in the data. For this exercise we consider data that stopped at the end of 2019, just before the Covid pandemic started to affect the economy, and we consider quarterly data over the full sample. In both of these settings, figures qualitatively similar to Figure 1 were obtained. In all four of these exercises, we used $\omega = 0.1$, and found γ and μ combinations roughly the same as in the primary investigation, with γ around 1.9 and μ around 0.2.

3.4 Comovements between the stochastic discount factor and risky returns

In this section we investigate a few other features of the model as a way of checking our fitting exercise. These features include: (i) investigating an alternative fitting approach that does not make use of $\rho_{m,R} = -0.2$, a common statistic used earlier to connect the Sharpe ratio to the Hansen-Jagannathan bound, and (ii) investigating the implied risk-free rate of return.

Our baseline fitting algorithm focused on fitting the Hansen-Jagannathan bound, which is linked to the observed Sharpe Ratio. This is a natural objective since the two concepts are at the center of much of the equity premium literature. An alternative fitting algorithm might focus on fitting $cov_t\left(m_{t+1}, R_{t+1}\right)$. Although not a central focus in the equity premium literature, this statistic does have some attractions. To understand these, note that a few steps of algebra away from the Sharpe ratio algebra, or expression (2), gives

$$\begin{aligned} E_t\left(R_{t+1}\right)-R_{t+1}^f &= -\frac{cov_t\left(m_{t+1}, R_{t+1}\right)}{E_t\left(m_{t+1}\right)}=-R_{t+1}^f cov_t\left(m_{t+1}, R_{t+1}\right) \qquad \text{or} \\ cov_t\left(m_{t+1}, R_{t+1}\right) &= -\frac{\left(E_t\left(R_{t+1}\right)-R_{t+1}^f\right)}{R_{t+1}^f}. \end{aligned}$$

This expression shows that $cov_t\left(m_{t+1}, R_{t+1}\right)$ is equal to the ratio between two widely discussed observables. In particular, the equity premium, $E_t\left(R_{t+1}\right)-R_{t+1}^f$, is estimated to be near 7%, while the risk-free return, R_{t+1}^f, is near 1 as seen in Table 2. Using these results, we considered a fitting algorithm similar to the one described above where we generated time series for m_{t+1} and R_{t+1} for a particular set of parameter values. We then ran GARCH(1,1) models on m_{t+1}, R_{t+1} and $m_{t+1}\times R_{t+1}$ to obtain conditional means for each of these series in the same way as we did earlier. These results were then used to compute $cov_t\left(m_{t+1}, R_{t+1}\right)=E_t\left(m_{t+1}R_{t+1}\right)-E_t\left(m_{t+1}\right)\times E_t\left(R_{t+1}\right)$ for a particular set of parameters. Using a similar grid search algorithm as described above, we search over the parameter space until we found parameters so that $cov_t\left(m_{t+1}, R_{t+1}\right)$ is roughly -0.07. The results of this exercise are shown in Figure 5.[23]

Table 2. Equity premium and risk-free rate

	Sample period		
	1889-1978	1934-2022	1955-2022
Mean of equity premium	0.0618	0.0838	0.0723
Mean of risk-free rate	1.008	0.9982	1.0061

Notes: The estimates for the sample period 1889-1978 are taken from Mehra (2003, Table

[23] In Figure 5, we plot $-1\times cov_t\left(m_{t+1}, R_{t+1}\right)$ to aid in the interpretation of the diagram.

4), while the statistics for the other two sample periods were obtained by the authors using the times series described above.

Looking at the left edge of Figure 5, where $\mu = 0$, which is the value in which the modified utility function equals the standard CRRA utility function, we see that the $cov_t(m_{t+1}, R_{t+1})$ is close to zero. In particular, for values of γ in the empirical range, the $cov_t(m_{t+1}, R_{t+1})$ never gets close to -0.06, the lower value of the range for the equity premium shown in Table 2. However, for $\gamma = 1.872$, and for values of μ that are not too large, in the range $0.23 - 0.24$, we see that it is possible to get the $cov_t(m_{t+1}, R_{t+1})$ in the range of 0.05 to 0.1. Furthermore, this range of parameter values are similar to those found solving the equity premium puzzle in Figure 1 above. Moreover, Figure 6 shows that for the same range of parameter values solving the equity premium puzzle, the inverse of the conditional mean of the stochastic discount factor implies a risk-free return around 1.00987, which is close to the value of 1.008 reported in Mehra (2008) for the period 1889-1978 and the values reported in Table 2 for alternative sample periods. In short, the same range of parameter values solving the equity premium puzzle does a good job in solving the risk-free puzzle coined by Weil (1989). Finally, Figure 7 shows the implied Hansen-Jagannathan bound implied by the simulated $cov_t(m_{t+1}, R_{t+1})$ for alternative values of γ and μ, which is rather similar to the displayed in Figure 1 obtained setting the conditional correlation between equity returns and the stochastic discount factor equal to -0.2. Together, these figures show that several important empirical observations can be simultaneously achieved with the modified utility model.

4 Conclusion

In this paper we argue that agents have a preference for consumption predictability. We find that by adding a desire for consumption predictability into the standard constant rate of relative risk aversion utility function we are able to solve the equity premium puzzle. The intuition for why this works is that the added interest in pre-

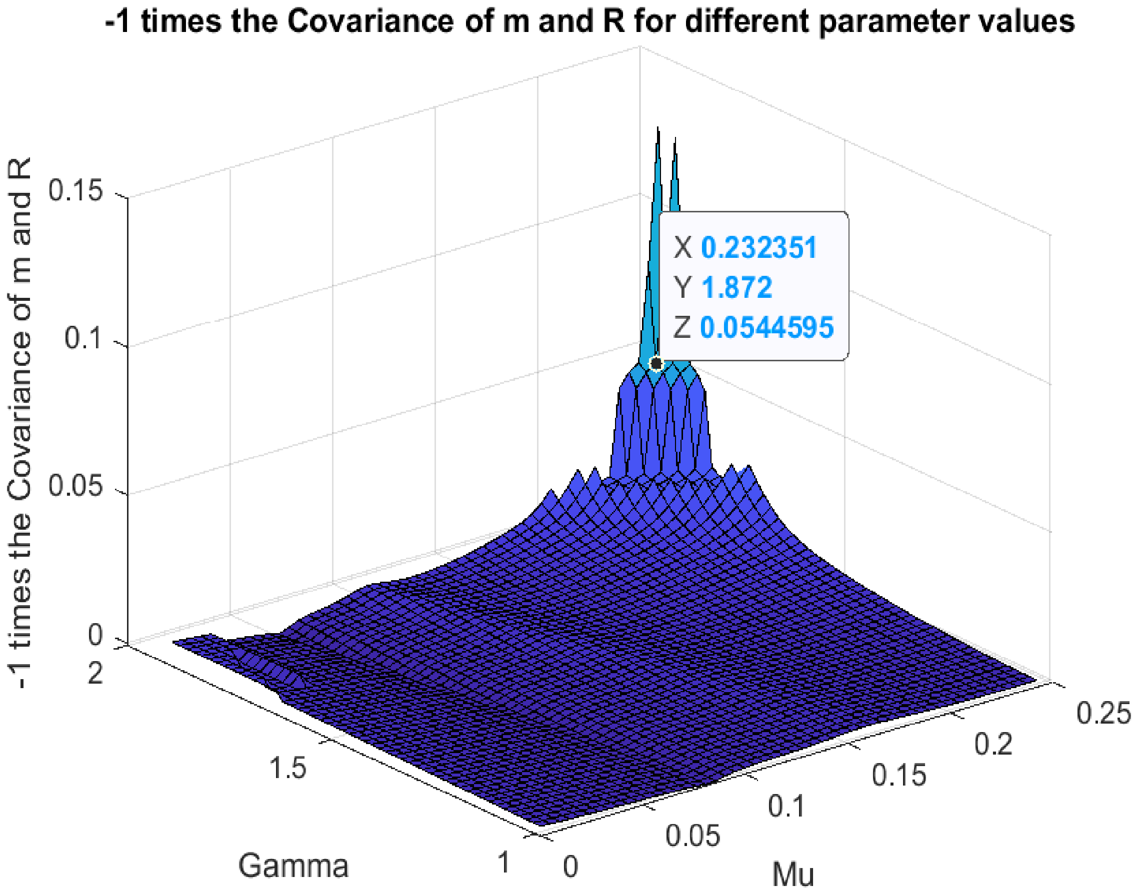

Figure 5: $(-1)\times$ Conditional covariance between the stochastic discont factor and the equity return

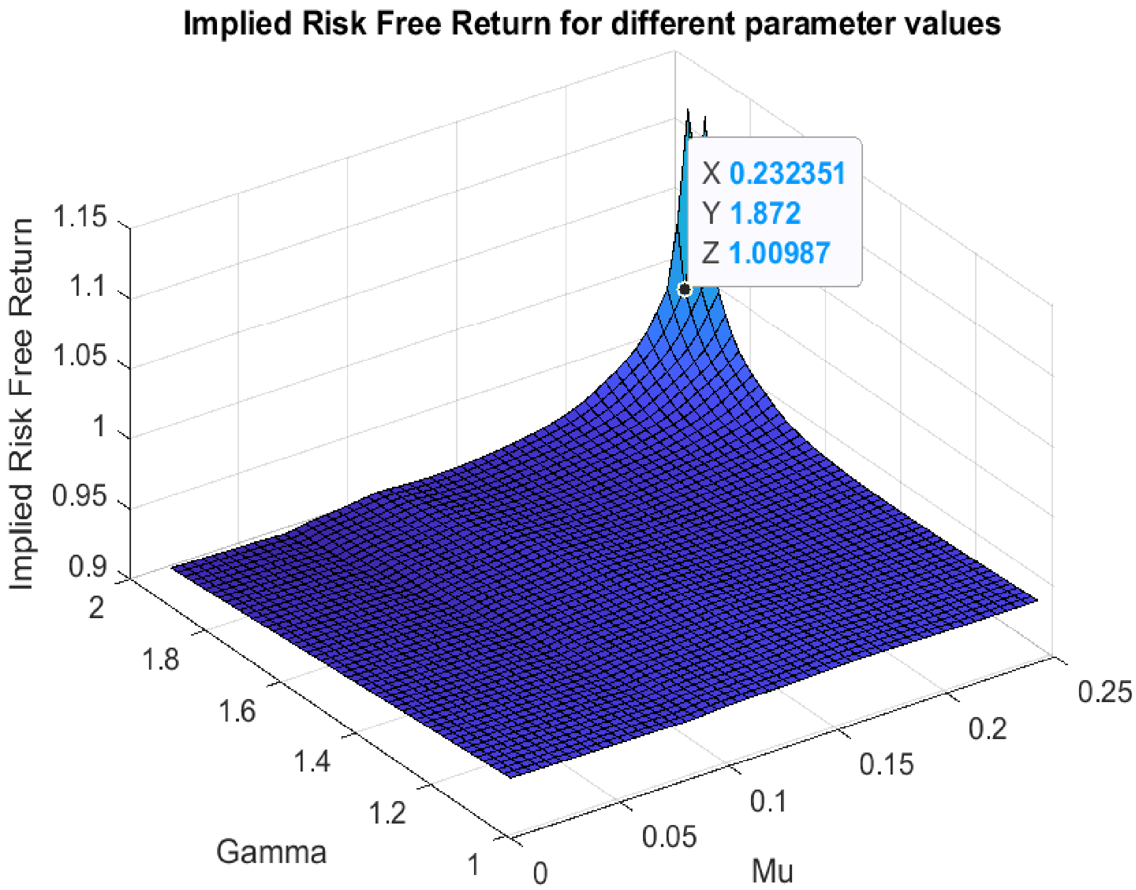

Figure 6: The inverse of the conditional mean of the stochastic discont factor

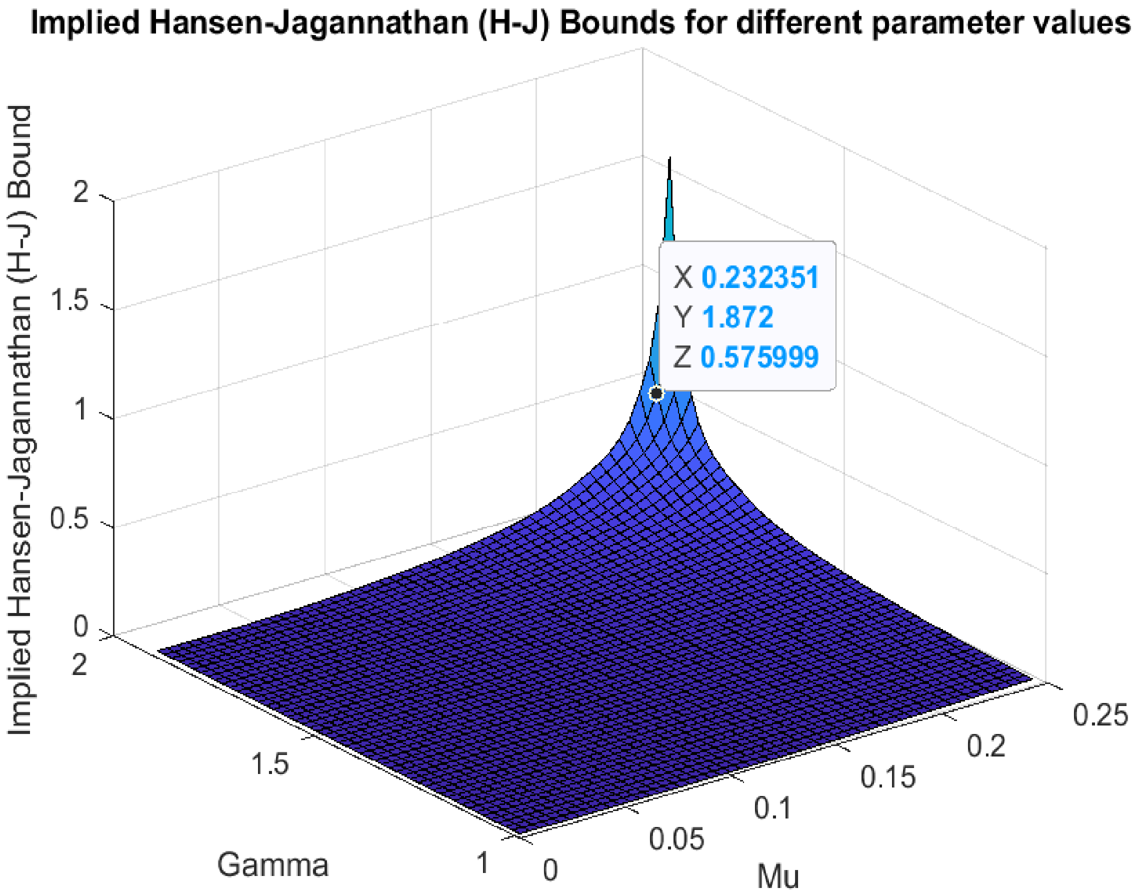

Figure 7: The Hansen-Jagannathan bound implied by the simulated $cov_t\left(m_{t+1}, R_{t+1}\right)$

dictability increases the desire to smooth consumption. Using observed consumption data, we find that there is enough unpredictability in the data to induce consumers to hold less risk despite the equity premium. We also sought to understand the mechanism more deeply and connected the result to increased volatility in the stochastic discount factor, a time varying rate of relative risk aversion (RRA) and a time varying intertemporal elasticity of substitution (IES). We find that the unpredictability of the observed consumption series results in increased volatility in the stochastic discount factor, and that during periods of extraordinary unpredictability, such as during recent economic contractions, agents time varying RRA becomes very high and their IES becomes very low as agents seek to smooth their consumption. In addition, it was found that an alternative fitting algorithm which focused on fitting the comovements of the stochastic discount factor and risky returns resulted in similar parameter values, demonstrating that the modified model is able to fit several different empirical observations simultaneously.

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5 Appendix 1: Details for mathematical calculations

In this appendix we provide some additional details for the mathematical expressions given earlier in the paper. This appendix is not intended for publication but rather help readers check the mathematics. We organize these into several subsections for clarity.

5.1 Lagrangian, first order conditions and the asset pricing equations

The representative agent chooses $\{c_t, b_{t+1} : t \geq 0\}$ so as to maximize

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\gamma} c_t^{1-\gamma} - \frac{\mu}{1-\gamma} \left(\omega c_t^2 + (c_t - E_{t-1}\{c_t\})^2 \right)^{\frac{1-\gamma}{2}} \right) \right\}$$

subject to $c_t + b_{t+1} = R_t b_t + w_t$ given b_0. One way of interpreting the forecast error term is to say that consumers have an extra smoothness term in their utility function. When forecast errors are small utility is higher and when forecast errors are large utility is lower.

One way of writing the Lagrangian for this problem is

$$\mathcal{L}(\cdot) = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{1-\gamma} c_t^{1-\gamma} - \frac{\mu}{1-\gamma} \left(\omega c_t^2 + (c_t - E_{t-1}\{c_t\})^2 \right)^{\frac{1-\gamma}{2}} + \lambda_t \left[R_t b_t + w_t - c_t - b_{t+1} \right] \right] \right\}.$$

The first order conditions for $t = 0, 1, \ldots$ are

$$\frac{\partial \mathcal{L}(\cdot)}{\partial c_t} : c_t^{-\gamma} - \mu \left(\omega c_t^2 + (c_t - E_{t-1}\{c_t\})^2 \right)^{\frac{1-\gamma}{2}-1} \left[\omega c_t + \left((c_t - E_{t-1}\{c_t\})^2 \right)^{\frac{1}{2}} \right] - \lambda_t = 0, \tag{3}$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial b_{t+1}} : E_t \left\{ \beta \lambda_{t+1} R_{t+1} - \lambda_t \right\} = 0, \tag{4}$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial \lambda_t} : R_t b_t + w_t - c_t - b_{t+1} = 0. \tag{5}$$

Solving (3) and substituting into (4) we get

$$E_t\left\{\begin{array}{c}\beta\left[c_{t+1}^{-\gamma}-\mu\left(\omega c_{t+1}^2+(c_{t+1}-E_t\{c_{t+1}\})^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_{t+1}+\left((c_{t+1}-E_t\{c_{t+1}\})^2\right)^{\frac{1}{2}}\right]\right]R_{t+1}\\ -\left[c_t^{-\gamma}-\mu\left(\omega c_t^2+(c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_t+\left((c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1}{2}}\right]\right]\end{array}\right\}=0.$$

or

$$E_t\left\{\begin{array}{c}\beta\left[c_{t+1}^{-\gamma}-\mu\left(\omega c_{t+1}^2+(\varepsilon_{t+1})^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_{t+1}+\left((\varepsilon_{t+1})^2\right)^{\frac{1}{2}}\right]\right]R_{t+1}\\ -\left[c_t^{-\gamma}-\mu\left(\omega c_t^2+(\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_t+\left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]\right]\end{array}\right\}=0.$$

Which can be rearranged to get

$$E_t\left\{\frac{\beta\left[c_{t+1}^{-\gamma}-\mu\left(\omega c_{t+1}^2+(\varepsilon_{t+1})^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_{t+1}+\left((\varepsilon_{t+1})^2\right)^{\frac{1}{2}}\right]\right]R_{t+1}}{\left[c_t^{-\gamma}-\mu\left(\omega c_t^2+(\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_t+\left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]\right]}\right\}=1,$$

or

$$\begin{aligned}E_t\left\{m_{t+1}R_{t+1}\right\} &= 1 \quad \text{where} \qquad (6)\\ m_{t+1} &= \frac{\beta\left[c_{t+1}^{-\gamma}-\mu\left(\omega c_{t+1}^2+(\varepsilon_{t+1})^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_{t+1}+\left((\varepsilon_{t+1})^2\right)^{\frac{1}{2}}\right]\right]}{\left[c_t^{-\gamma}-\mu\left(\omega c_t^2+(\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_t+\left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]\right]}.\end{aligned}$$

One could imagine a risk-free asset in the model. In this case

$$\begin{aligned}E_t\left\{m_{t+1}\right\}R_{t+1}^f &= 1 \quad \text{where again} \qquad (7)\\ m_{t+1} &= \frac{\beta\left[c_{t+1}^{-\gamma}-\mu\left(\omega c_{t+1}^2+(\varepsilon_{t+1})^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_{t+1}+\left((\varepsilon_{t+1})^2\right)^{\frac{1}{2}}\right]\right]}{\left[c_t^{-\gamma}-\mu\left(\omega c_t^2+(\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_t+\left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]\right]}.\end{aligned}$$

5.2 The Hansen-Jagannathan lower bound

Equation (6) can be written as

$$\begin{aligned}1 &= E_t\left(m_{t+1}R_{t+1}\right)=E_t\left(m_{t+1}\right)E_t\left(R_{t+1}\right)+cov_t\left(m_{t+1},R_{t+1}\right)\\ &= E_t\left(m_{t+1}\right)E_t\left(R_{t+1}\right)+\rho_{m,R}\left[var_t\left(m_{t+1}\right)var_t\left(R_{t+1}\right)\right]^{\frac{1}{2}},\end{aligned}$$

where $\rho_{m,R}$ is the *conditional* correlation coefficient between m_{t+1} and R_{t+1}. To find a formula for the excess return, $E_t(R_{t+1}) - R^f_{t+1}$, rewrite this as

$$\begin{aligned} E_t(m_{t+1})\,E_t(R_{t+1}) - 1 &= -\rho_{m,R}\left[var_t(m_{t+1})\,var_t(R_{t+1})\right]^{\frac{1}{2}} \\ E_t(m_{t+1})\,E_t(R_{t+1}) - E_t\{m_t\}\,R^f_{t+1} &= -\rho_{m,R}\left[var_t(m_{t+1})\,var_t(R_{t+1})\right]^{\frac{1}{2}} \qquad \text{Here we use (7).} \\ E_t(R_{t+1}) - R^f_{t+1} &= -\rho_{m,R}\frac{std_t(m_{t+1})}{E_t(m_{t+1})}std_t(R_{t+1}). \qquad (8) \end{aligned}$$

Note, equation (8) is equation (2) referenced at the end of section 2.1 which we note is, "one step prior to the Hansen-Jagannathan equation". Since (the absolute value of) correlation coefficients cannot be greater than 1, then

$$\left|E_t(R_{t+1}) - R^f_{t+1}\right| \leq \frac{std_t(m_{t+1})}{E_t(m_{t+1})}std_t(R_{t+1}),$$

or

$$\left|\frac{E_t(R_{t+1}) - R^f_{t+1}}{std_t(R_{t+1})}\right| \leq \frac{std_t(m_{t+1})}{E_t(m_{t+1})}$$

which is the standard Hansen-Jagannathan lower bound equation.

5.3 The Rate of Relative Risk Aversion

Define the consumption marginal utility by

$$CMU_t = c_t^{-\gamma} - \mu\left(\omega c_t^2 + (c_t - E_{t-1}\{c_t\})^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_t + \left((c_t - E_{t-1}\{c_t\})^2\right)^{\frac{1}{2}}\right].$$

Then we have the following derivative, $\frac{\partial CMU_t}{\partial c_t} =$

$$
\begin{aligned}
& -\gamma c_t^{-\gamma-1} \\
-\mu & \left[\begin{array}{c} \left(\frac{1-\gamma}{2}-1\right)\left(\omega c_t^2+(c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1-\gamma}{2}-2}\left[\omega(2)c_t+(2)\left|c_t-E_{t-1}\{c_t\}\right|\right]\left[\omega c_t+\left((c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1}{2}}\right]+ \\ \left(\omega c_t^2+(c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1-\gamma}{2}-1}\left(\omega+\frac{1}{2}\left((c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1}{2}-1}(2)\left|c_t-E_{t-1}\{c_t\}\right|(1)\right) \end{array} \right] \\
= & -\gamma c_t^{-\gamma-1} \\
-\mu & \left[\begin{array}{c} \left(\frac{1-\gamma}{2}-1\right)\left(\omega c_t^2+(c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1-\gamma}{2}-2}(2)\left[\omega c_t+\left((c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1}{2}}\right]\left[\omega c_t+\left((c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1}{2}}\right]+ \\ \left(\omega c_t^2+(c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1-\gamma}{2}-1}\left(\omega+\left((c_t-E_{t-1}\{c_t\})^2\right)^{0}\right) \end{array} \right] \\
= & -\gamma c_t^{-\gamma-1} \\
-\mu & \left[\begin{array}{c} \left(\frac{1-\gamma}{2}-1\right)\left(\omega c_t^2+(c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1-\gamma}{2}-2}(2)\left[\omega c_t+\left((c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1}{2}}\right]^2+ \\ \left(\omega c_t^2+(c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1-\gamma}{2}-1}(\omega+1) \end{array} \right] \\
= & -\gamma c_t^{-\gamma-1} \\
-\mu & \left[(1-\gamma-2)\left(\omega c_t^2+(c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1-\gamma}{2}-2}\left[\omega c_t+\left((c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1}{2}}\right]^2+\left(\omega c_t^2+(c_t-E_{t-1}\{c_t\})^2\right)^{\frac{1-\gamma}{2}-1}(\omega+1)\right] \\
= & -\gamma c_t^{-\gamma-1} \\
-\mu & \left[(-\gamma-1)\left(\omega c_t^2+(\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-2}\left[\omega c_t+\left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]^2+\left(\omega c_t^2+(\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}(\omega+1)\right] \\
= & -\gamma c_t^{-\gamma-1} \\
+\mu & \left[(\gamma+1)\left(\omega c_t^2+(\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-2}\left[\omega c_t+\left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]^2-\left(\omega c_t^2+(\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}(\omega+1)\right]
\end{aligned}
$$

Then the rate of relative risk aversion is given by

$$
\begin{aligned}
RRA_t &= -\frac{\frac{\partial CMU_t}{\partial c_t}}{CMU_t}c_t \\
&= \frac{\gamma c_t^{-\gamma-1}-\mu\left[(\gamma+1)\left(\omega c_t^2+(\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-2}\left[\omega c_t+\left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]^2-\left(\omega c_t^2+(\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}(\omega+1)\right]}{c_t^{-\gamma}-\mu\left(\omega c_t^2+(\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_t+\left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]}\times c_t
\end{aligned}
$$

Notice that for $\mu = 0$, one obtains the well-know result:

$$RRA_t = \gamma,$$

5.4 The Elasticity of Substitution

Let's start by looking at the definition of IES. It is given by the following.

$$IES_{t+1} = \frac{\partial \frac{c_{t+1}}{c_t}}{\partial \frac{p_t}{p_{t+1}}} \times \frac{\frac{p_t}{p_{t+1}}}{\frac{c_{t+1}}{c_t}} = \frac{\partial \frac{c_{t+1}}{c_t}}{\partial R_{t+1}} \times \frac{R_{t+1}}{\frac{c_{t+1}}{c_t}}.$$

where $\frac{p_t}{p_{t+1}} = R_{t+1}$. The fraction on the right side of the $\times$ is straightforward, but the derivative term is hard. Recall that

$$E_t \left\{ \frac{\beta \left[c_{t+1}^{-\gamma} - \mu \left(\omega c_{t+1}^2 + (\varepsilon_{t+1})^2 \right)^{\frac{1-\gamma}{2}-1} \left[\omega c_{t+1} + \left((\varepsilon_{t+1})^2 \right)^{\frac{1}{2}} \right] \right] R_{t+1}}{\left[c_t^{-\gamma} - \mu \left(\omega c_t^2 + (\varepsilon_t)^2 \right)^{\frac{1-\gamma}{2}-1} \left[\omega c_t + \left((\varepsilon_t)^2 \right)^{\frac{1}{2}} \right] \right]} \right\} = 1,$$

There are several standard things to note. First, the denominator is a constant based on information at time t so it can be multiplied through. Second the expectation terms in the forecast error are also constants, so derivatives with respect to those terms are zero. Let's rearrange this by cross multiplying the denominator and dividing by $c_t^{-\gamma}$ which is done in several steps. To keep things simple, after the second step, the right side is just labeled constant because when taking the derivative it will be zero.

$$E_t \left\{ \beta \left[c_{t+1}^{-\gamma} - \mu \left(\omega c_{t+1}^2 + (\varepsilon_{t+1})^2 \right)^{\frac{1-\gamma}{2}-1} \left[\omega c_{t+1} + \left((\varepsilon_{t+1})^2 \right)^{\frac{1}{2}} \right] \right] R_{t+1} \right\} = c_t^{-\gamma} - \mu \left(\omega c_t^2 + (\varepsilon_t)^2 \right)^{\frac{1-\gamma}{2}-1} \left[\omega c_t + \left((\varepsilon_t)^2 \right)^{\frac{1}{2}} \right]$$

$$\frac{E_t \left\{ \beta \left[c_{t+1}^{-\gamma} - \mu \left(\omega c_{t+1}^2 + (\varepsilon_{t+1})^2 \right)^{\frac{1-\gamma}{2}-1} \left[\omega c_{t+1} + \left((\varepsilon_{t+1})^2 \right)^{\frac{1}{2}} \right] \right] R_{t+1} \right\}}{c_t^{-\gamma}} = \frac{c_t^{-\gamma} - \mu \left(\omega c_t^2 + (\varepsilon_t)^2 \right)^{\frac{1-\gamma}{2}-1} \left[\omega c_t + \left((\varepsilon_t)^2 \right)^{\frac{1}{2}} \right]}{c_t^{-\gamma}}$$

$$E_t \left\{ \beta \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} - \mu \left(\omega c_{t+1}^2 + (\varepsilon_{t+1})^2 \right)^{\frac{-\gamma-1}{2}} \left(\frac{1}{c_t^{-\gamma-1}} \right) \left(\frac{1}{c_t} \right) \left[\omega c_{t+1} + \left((\varepsilon_{t+1})^2 \right)^{\frac{1}{2}} \right] \right] R_{t+1} \right\} = \text{constant.}$$

$$E_t \left\{ \beta \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} - \mu \left(\omega \left(\frac{c_{t+1}}{c_t} \right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t} \right)^2 \right)^{\frac{-\gamma-1}{2}} \left[\omega \left(\frac{c_{t+1}}{c_t} \right) + \left(\left(\frac{\varepsilon_{t+1}}{c_t} \right)^2 \right)^{\frac{1}{2}} \right] \right] R_{t+1} \right\} = \text{constant.}$$

Totally differentiating where we keep the $\frac{c_{t+1}}{c_t}$ terms together and compute $\partial \frac{c_{t+1}}{c_t}$ and recognizing that the term $\frac{E_t\{c_{t+1}\}}{c_t}$ is a constant since both the numerator and denominator are constants at time t and noting that

$$\frac{\partial \frac{\varepsilon_{t+1}}{c_t}}{\partial \frac{c_{t+1}}{c_t}} = \frac{\partial \frac{c_{t+1} - E_t\{c_{t+1}\}}{c_t}}{\partial \frac{c_{t+1}}{c_t}} = \frac{\partial \left(\frac{c_{t+1}}{c_t} - \frac{E_t\{c_{t+1}\}}{c_t} \right)}{\partial \frac{c_{t+1}}{c_t}} = 1 - 0 = 1,$$

We get

$$E_t\left\{\begin{array}{c}\beta\left[\begin{array}{c}-\gamma\left(\frac{c_{t+1}}{c_t}\right)^{-(\gamma+1)}\left(\partial\frac{c_{t+1}}{c_t}\right)\\ -\mu\left(\frac{-\gamma-1}{2}\right)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}-1}(2)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)+\left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right)\left(\partial\frac{c_{t+1}}{c_t}\right)\left[\omega\left(\frac{c_{t+1}}{c_t}\right)+\left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]\\ -\mu\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}(\omega+1)\left(\partial\frac{c_{t+1}}{c_t}\right)\end{array}\right]R^f_{t+1}\\ +\beta\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}-\mu\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\left[\omega\left(\frac{c_{t+1}}{c_t}\right)+\left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]\right]\partial R^f_{t+1}\end{array}\right.$$

=0. Simplifying

$$E_t\left\{\begin{array}{c}\beta\left[\begin{array}{c}-\gamma\left(\frac{c_{t+1}}{c_t}\right)^{-(\gamma+1)}\left(\partial\frac{c_{t+1}}{c_t}\right)\\ +\mu(\gamma+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}-1}\left[\omega\left(\frac{c_{t+1}}{c_t}\right)+\left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]^2\left(\partial\frac{c_{t+1}}{c_t}\right)\\ -\mu(\omega+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\left(\partial\frac{c_{t+1}}{c_t}\right)\end{array}\right]R^f_{t+1}\\ +\beta\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}-\mu\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\left[\omega\left(\frac{c_{t+1}}{c_t}\right)+\left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]\right]\partial R^f_{t+1}\end{array}\right\}=0,$$

Removing the expectation and cross multiplying

$$\begin{aligned}&\beta\left[\begin{array}{c}-\gamma\left(\frac{c_{t+1}}{c_t}\right)^{-(\gamma+1)}\\ +\mu(\gamma+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}-1}\left[\omega\left(\frac{c_{t+1}}{c_t}\right)+\left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]^2\\ -\mu(\omega+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\end{array}\right]\left(\partial\frac{c_{t+1}}{c_t}\right)R^f_{t+1}\\ =\ &-\beta\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}-\mu\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\left[\omega\left(\frac{c_{t+1}}{c_t}\right)+\left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]\right]\partial R^f_{t+1}.\end{aligned}$$

Eliminating the $-\beta$ gives

$$\begin{aligned}&\left[\begin{array}{c}\gamma\left(\frac{c_{t+1}}{c_t}\right)^{-(\gamma+1)}\\ -\mu(\gamma+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}-1}\left[\omega\left(\frac{c_{t+1}}{c_t}\right)+\left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]^2\\ +\mu(\omega+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\end{array}\right]\left(\partial\frac{c_{t+1}}{c_t}\right)R^f_{t+1}\\ =\ &\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}-\mu\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2+\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\left[\omega\left(\frac{c_{t+1}}{c_t}\right)+\left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]\right]\partial R^f_{t+1}.\end{aligned}$$

Now solving for $\frac{\partial \frac{c_{t+1}}{c_t}}{\partial R^f_{t+1}}$ we get

$$\frac{\partial \frac{c_{t+1}}{c_t}}{\partial R^f_{t+1}} = \frac{\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} - \mu\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\left[\omega\left(\frac{c_{t+1}}{c_t}\right) + \left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]\right]}{\left[\begin{array}{c}\gamma\left(\frac{c_{t+1}}{c_t}\right)^{-(\gamma+1)} \\ -\mu(\gamma+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}-1}\left[\omega\left(\frac{c_{t+1}}{c_t}\right) + \left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]^2 \\ +\mu(\omega+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\end{array}\right] R^f_{t+1}}$$

Plugging this in the IES formula to get

$$\begin{aligned} IES_{t+1} &= \frac{\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} - \mu\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\left[\omega\left(\frac{c_{t+1}}{c_t}\right) + \left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]\right]}{\left[\begin{array}{c}\gamma\left(\frac{c_{t+1}}{c_t}\right)^{-(\gamma+1)} \\ -\mu(\gamma+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}-1}\left[\omega\left(\frac{c_{t+1}}{c_t}\right) + \left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]^2 \\ +\mu(\omega+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\end{array}\right] R^f_{t+1}} \times \frac{R^f_{t+1}}{\frac{c_{t+1}}{c_t}} \\ &= \frac{\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} - \mu\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\left[\omega\left(\frac{c_{t+1}}{c_t}\right) + \left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]\right]}{\left[\begin{array}{c}\gamma\left(\frac{c_{t+1}}{c_t}\right)^{-(\gamma+1)} \\ -\mu(\gamma+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}-1}\left[\omega\left(\frac{c_{t+1}}{c_t}\right) + \left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]^2 \\ +\mu(\omega+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\end{array}\right]} \times \left(\frac{c_{t+1}}{c_t}\right)^{-1}. \end{aligned}$$

5.5 Verification that $IES_{t+1} = 1/RRA_{t+1}$

An interesting question is whether the ISE_{t+1} is the reciprocal of the RRA_{t+1} as it is in the standard CRRA model. Let's investigate this. The

$$RRA_t = \frac{\left[\gamma c_t^{-\gamma-1} - \mu\left[(\gamma+1)\left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-2}\left[\omega c_t + \left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]^2 - \left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}(\omega+1)\right]\right]}{c_t^{-\gamma} - \mu\left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_t + \left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]} \times c_t$$

so

$$\frac{1}{RRA_t} = \frac{c_t^{-\gamma} - \mu\left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_t + \left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]}{\left[\gamma c_t^{-\gamma-1} - \mu\left[(\gamma+1)\left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-2}\left[\omega c_t + \left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right]^2 - \left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}(\omega+1)\right]\right] c_t}$$

Incrementing everything by one period gives $\frac{1}{RRA_{t+1}} =$

$$\frac{c_{t+1}^{-\gamma} - \mu\left(\omega c_{t+1}^2 + (\varepsilon_{t+1})^2\right)^{\frac{1-\gamma}{2}-1}\left[\omega c_{t+1} + \left((\varepsilon_{t+1})^2\right)^{\frac{1}{2}}\right]}{\left[\gamma c_{t+1}^{-\gamma-1} - \mu\left[(\gamma+1)\left(\omega c_{t+1}^2 + (\varepsilon_{t+1})^2\right)^{\frac{1-\gamma}{2}-2}\left[\omega c_{t+1} + \left((\varepsilon_{t+1})^2\right)^{\frac{1}{2}}\right]^2 - \left(\omega c_{t+1}^2 + (\varepsilon_{t+1})^2\right)^{\frac{1-\gamma}{2}-1}(\omega+1)\right]\right]}\left(\frac{1}{c_{t+1}}\right)$$

Multiplying by $\frac{\frac{1}{(c_t)^{-(1+\gamma)}}}{\frac{1}{(c_t)^{-(1+\gamma)}}} = \frac{\left(\frac{1}{c_t^{-\gamma}}\right)(c_t)}{\frac{1}{(c_t)^{-(1+\gamma)}}}$ gives $\frac{1}{RRA_{t+1}} =$

$$\frac{\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} - \mu\left(\omega c_{t+1}^2 + (\varepsilon_{t+1})^2\right)^{\frac{-\gamma-1}{2}}\left(\frac{1}{c_t^{-\gamma-1}}\right)\left(\frac{1}{c_t}\right)\left[\omega c_{t+1} + \left((\varepsilon_{t+1})^2\right)^{\frac{1}{2}}\right]\left(\frac{c_{t+1}}{c_t}\right)^{-1}}{\left[\begin{array}{c}\gamma\left(\frac{c_{t+1}}{c_t}\right)^{-(\gamma+1)} \\ -\mu\left[(\gamma+1)\left(\omega c_{t+1}^2 + (\varepsilon_{t+1})^2\right)^{\frac{1-\gamma-4}{2}}\left(\frac{1}{c_t^{-\gamma-3}}\right)\left(\frac{1}{c_t}\right)^2\left[\omega c_{t+1} + \left((\varepsilon_{t+1})^2\right)^{\frac{1}{2}}\right]^2 - \left(\frac{1}{c_t^{-\gamma-1}}\right)\left(\omega c_{t+1}^2 + (\varepsilon_{t+1})^2\right)^{\frac{-\gamma-1}{2}}(\omega+1)\right]\end{array}\right]}$$

$=$

$$\frac{\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} - \mu\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\left[\omega\frac{c_{t+1}}{c_t} + \left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]\left(\frac{c_{t+1}}{c_t}\right)^{-1}}{\left[\begin{array}{c}\gamma\left(\frac{c_{t+1}}{c_t}\right)^{-(\gamma+1)} \\ -\mu\left[(\gamma+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-3}{2}}\left[\omega\frac{c_{t+1}}{c_t} + \left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]^2 - \left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}(\omega+1)\right]\end{array}\right]}$$

$=$

$$\frac{\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} - \mu\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\left[\omega\frac{c_{t+1}}{c_t} + \left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]\left(\frac{c_{t+1}}{c_t}\right)^{-1}}{\left[\begin{array}{c}\gamma\left(\frac{c_{t+1}}{c_t}\right)^{-(\gamma+1)} \\ -\mu(\gamma+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}-1}\left[\omega\frac{c_{t+1}}{c_t} + \left(\left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{1}{2}}\right]^2 - \mu(\omega+1)\left(\omega\left(\frac{c_{t+1}}{c_t}\right)^2 + \left(\frac{\varepsilon_{t+1}}{c_t}\right)^2\right)^{\frac{-\gamma-1}{2}}\end{array}\right]}$$

$= IES_{t+1}$

5.6 Discussion on the sign of $\partial CMU/\partial \varepsilon_t$

Here we show that consumption and forecast errors are substitutes for standard values of consumption and consumption forecast errors where the former is much larger than the latter. That is, the marginal utility of consumption is, in general, a decreasing function of the consumption forecast error, ε_t. Let us use this notation to write the consumption marginal utility:

$$CMU_t = c_t^{-\gamma} - \mu \left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1} \left[\omega c_t + \left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right].$$

Then,

$$\begin{aligned}
\frac{\partial CMU}{\partial \varepsilon_t} &= -\mu \left[\omega c_t + \left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right] \left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-2} \left(\frac{1-\gamma}{2} - 1\right) 2\varepsilon_t - \mu \left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1} \\
&= -\mu \left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1} \left\{- \left[\omega c_t + \left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right] \left(\omega c_t^2 + (\varepsilon_t)^2\right)^{-1} (1+\gamma)\varepsilon_t + 1\right\} \\
&= -\mu \frac{\left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}}{\left(\omega c_t^2 + (\varepsilon_t)^2\right)} \left\{- \left[\omega c_t + \left((\varepsilon_t)^2\right)^{\frac{1}{2}}\right] (1+\gamma)\varepsilon_t + \left(\omega c_t^2 + (\varepsilon_t)^2\right)\right\} \\
&= -\mu \frac{\left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}}{\left(\omega c_t^2 + (\varepsilon_t)^2\right)} \left\{- \left[\omega c_t \varepsilon_t + (\varepsilon_t)^2\right] (1+\gamma) + \left(\omega c_t^2 + (\varepsilon_t)^2\right)\right\} \\
&= -\mu \frac{\left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}}{\left(\omega c_t^2 + (\varepsilon_t)^2\right)} \left\{- \left[\omega c_t \varepsilon_t + (\varepsilon_t)^2 + \gamma \omega c_t \varepsilon_t + \gamma (\varepsilon_t)^2\right] + \left(\omega c_t^2 + (\varepsilon_t)^2\right)\right\} \\
&= -\mu \frac{\left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}}{\left(\omega c_t^2 + (\varepsilon_t)^2\right)} \left\{- \left[\omega c_t \varepsilon_t + \gamma \omega c_t \varepsilon_t + \gamma (\varepsilon_t)^2\right] + \omega c_t^2\right\} \\
&= -\mu \frac{\left(\omega c_t^2 + (\varepsilon_t)^2\right)^{\frac{1-\gamma}{2}-1}}{\left(\omega c_t^2 + (\varepsilon_t)^2\right)} \left\{\omega c_t \left[c_t - (1+\gamma)\, \varepsilon_t\right] - \gamma (\varepsilon_t)^2\right\}.
\end{aligned}$$

The sign of this partial derivative is negative whenever forecasts errors are negative (i.e. when consumption is lower than expected). Moreover, the sign would be still negative in the usual case when consumption is much larger than a positive forecast error. Put it differently, CMU only increases with huge, positive forecast errors. As an additional check, for the values of γ and ω that solve the equity premium puzzle one can check whether the time series of $\left\{\omega c_t \left[c_t - (1+\gamma)\,\varepsilon_t\right] - \gamma\,(\varepsilon_t)^2\right\}$ contains any negative number.

6 Appendix 2: Robustness checks

This appendix provides diagrams analogous to Figure 1 in the paper for a few different investigations. These investigations were briefly discussed in Section 3.3 Robustness. They include two different forecasting models and two different data structures. The different forecasting models are an AR(1) and an AR(2) of differenced consumption and the different data structures are a pre-Covid 19 annual data set and a quarterly full sample data set. This appendix is not intended to be published.

6.1 Forecasting model AR(1)

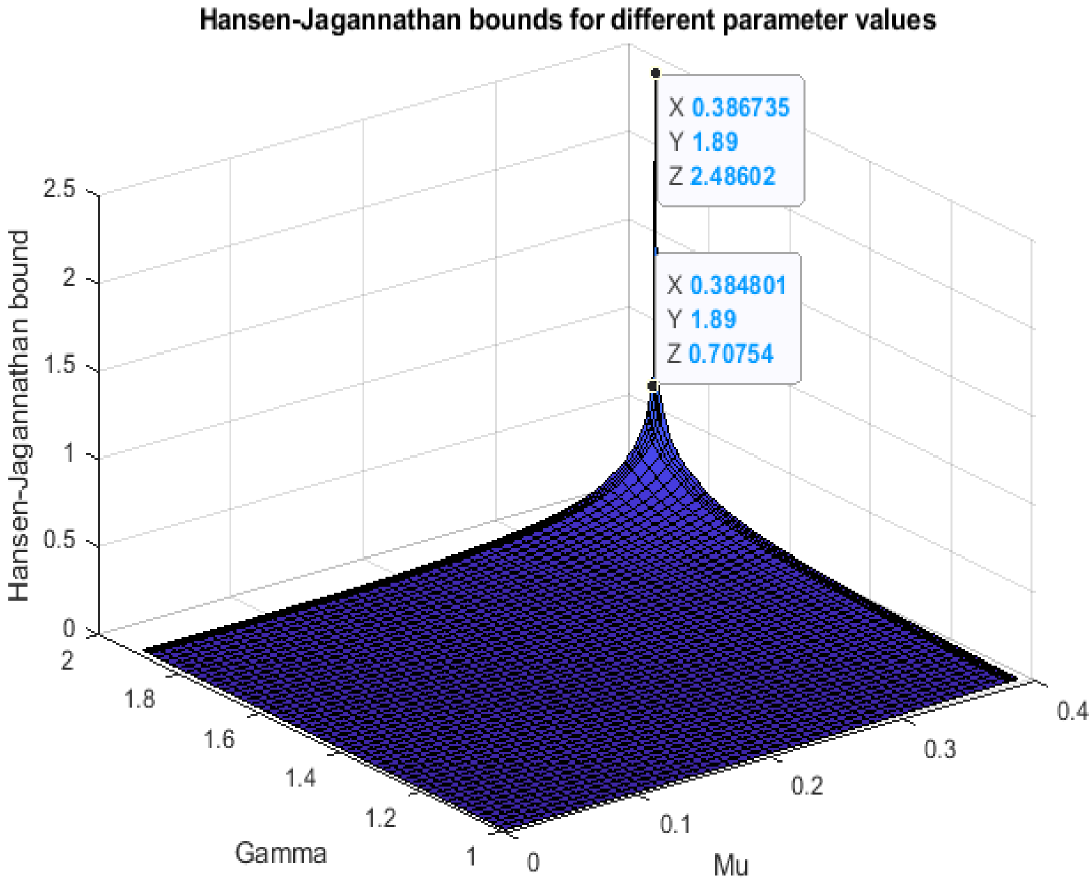

Figure A.1: AR(1) forecasting model

6.2 Forecasting model AR(2)

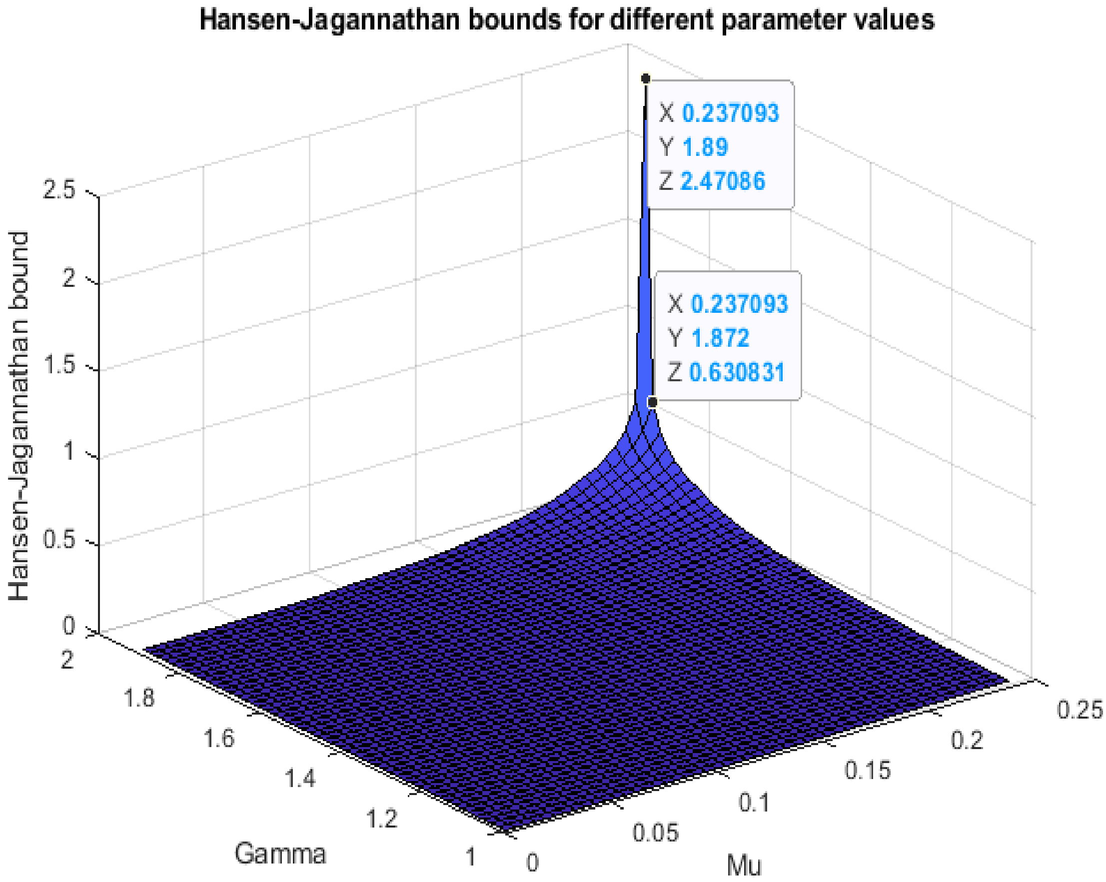

Figure A.2: AR(2) forecasting model

6.3 Pre-Covid sample (1955-2019)

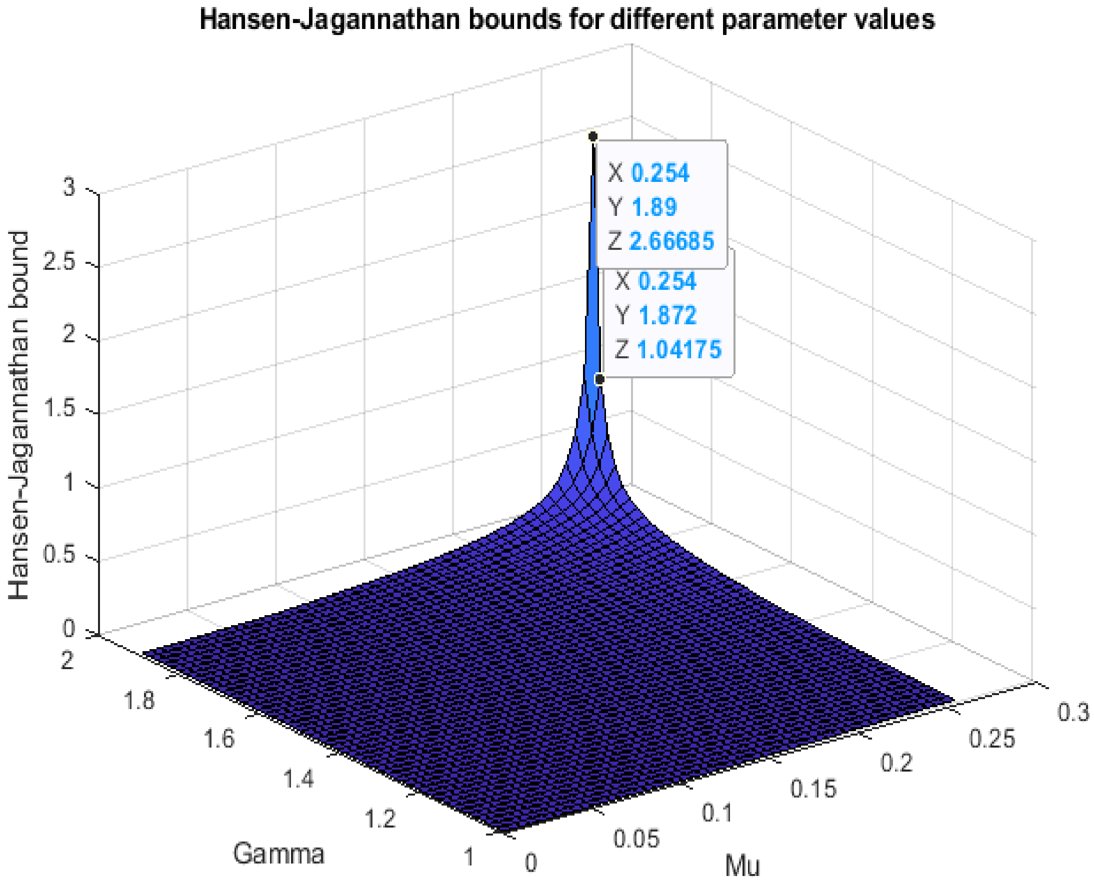

Figure A.3: Pre-Covid sample period

6.3.1 Quarterly sample period (1955:q1-2023:q1)

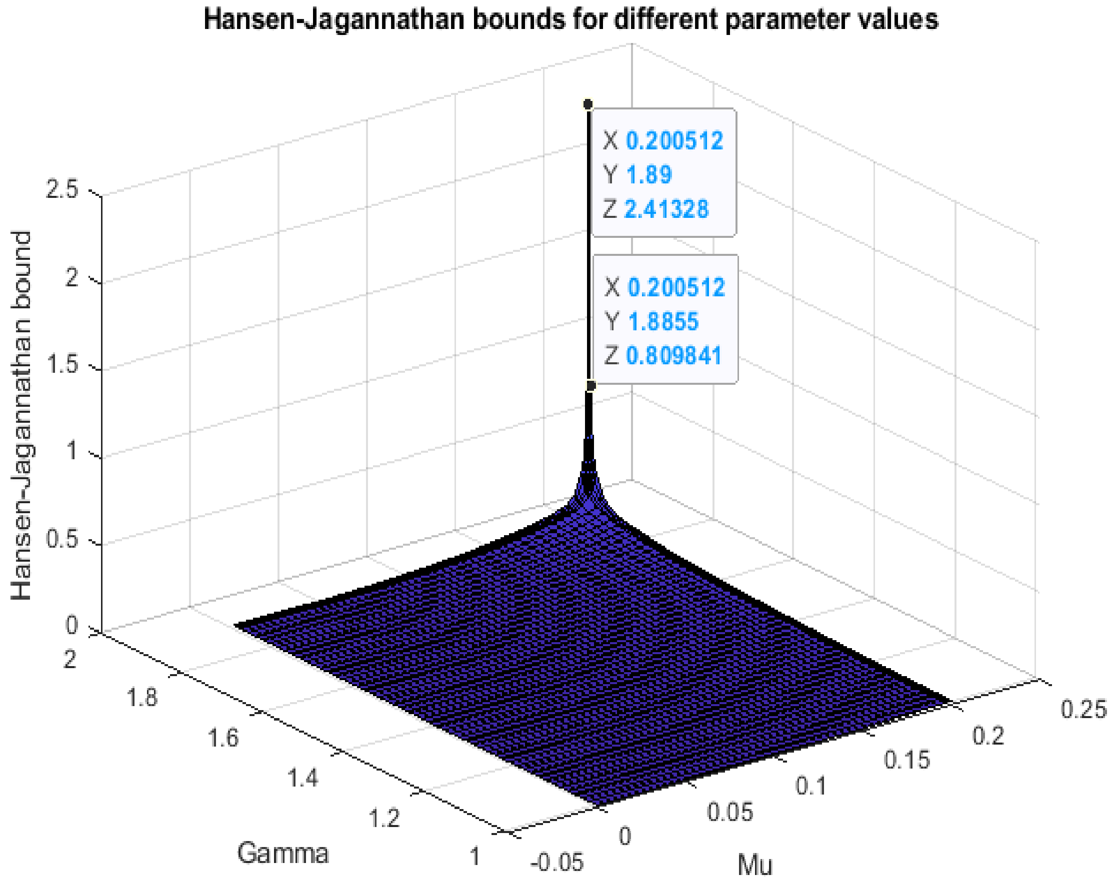

Figure A.4: Quarterly data frequency model